# Deep Dives into Natural Convection: Mantle Convection and Rayleigh-Bénard convection

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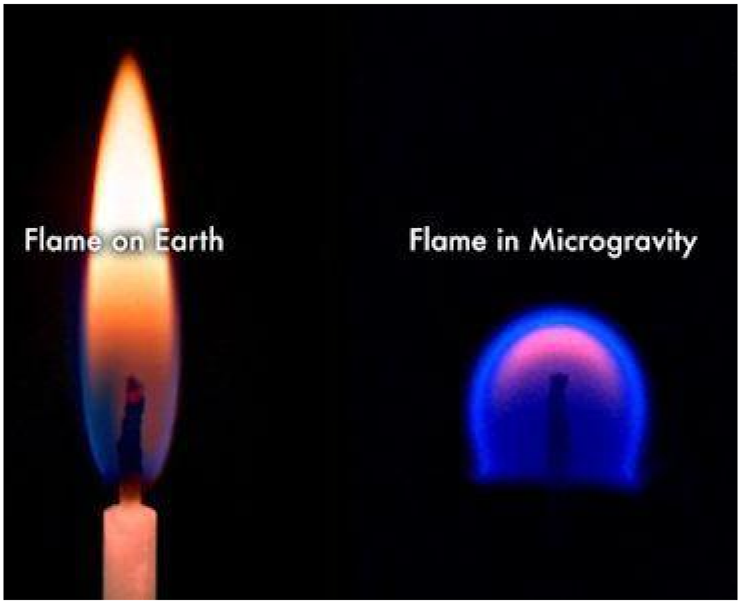
## Introduction: Free Convection vs Forced Convection

Natural or free convection is a type of heat and mass transport in which fluid motion is generated only by density differences between fluids, for example, a density difference occurring due to temperature change. Unlike forced convection, natural convection is not driven by external forces such as a pump, vacuum, or suction force[NPow]. Natural convection can be observed in many common processes such as the hot air rising in thermal updrafts that help birds fly, and the heat being carried away from a slice of pizza as it cools[Storey].

The general process for natural convection is as follows: fluid receives heat, thermal expansion occurs, fluid becomes less dense, heated fluid rises. The creation of convection currents relies on three principal assumptions[NPow].

1. Presence of a heat source - convection currents generated by density differences created by temperature gradients
2. Presence of proper acceleration - gravitational field, acceleration, centrifugal force, etc.
    a. Free convection cannot happen without acceleration, for example, a candle will not fully light in space because combustion of a candle wick requires free convection to supply to flame with air[NASA] .




[Fig.1 Flame on Earth (left) and flame in microgravity (right) NASA]

3. Proper geometry - allows for natural convection and is solely thermal conduction
    a. For example, if the temperature gradient goes in the same direction as the acceleration, there will only be conduction and no convection.

## Natural Convection Properties

The principle difference in motion equations for natural convection is that density is temperature dependent. We still ignore viscosity and thermal conductivity changes with temperature. As with any model, we use the given conditions to make a series of assumptions in order to approximate the flow characteristics. With the Boussinesq approximation, which applies in most cases, we assume the change in density from nominal value $\varrho_\infty$ is small relative to the total density. This density can be represented as:

$$\rho = \rho_\infty\left(1 + \beta\left(T - T_\infty\right)\right)$$ **Density for natural convection**

Where *T* is the temperature in Kelvin and $\beta$is the volumetric expansion coefficient (a property of the material), which for gas is $\beta = \frac{1}{T}$. If the change in density due to heating is small, we can assume a variable density only matters in the gravitational body force and thus variation with temperature is negligible. Incorporating these assumptions, the equations become:

$$\nabla \cdot v = 0$$

$$\rho_\infty \frac{Dv}{Dt} = -\nabla(P + \rho_\infty g z) + \rho_\infty \beta(T - T_\infty)g + \mu\nabla^2 v$$

$\rho_\infty c_p \frac{DT}{Dt} = k\nabla^2 T$

Of note, there is a force within the momentum equation that is proportional to the temperature, thus incorporating the changing temperature causing less dense air to rise (through the buoyant force). These equations can be made dimensionless as shown below:

$\nabla \cdot \overline{v} = 0$

$\frac{D\overline{v}}{D\overline{t}} = \ - \nabla\overline{P}_d + \frac{\beta g(T_s - T_\infty)\ell}{U_0^2}\overline{T} + \frac{1}{Re}\nabla^2\overline{v}$

$\frac{DT}{Dt} = \frac{1}{RePr}\nabla^2 T$

We choose $U_0$ to be $U_0 = \sqrt{\beta g(T_s - T_\infty)\ell}$ and can reduce the final momentum equation to:

$\frac{D\overline{v}}{D\overline{t}} = \ - \nabla\overline{P}_d + \overline{T} + \frac{1}{Re}\nabla^2\overline{v}$ **Momentum equation**

We make use of the Reynolds number, $Re$, which is defined in natural convection as Re = $\sqrt{Gr}$ where 'Gr' is the Grashof number. The Grashof number is the ratio of buoyant to viscous forces and is defined as having a driven velocity field. The Grashof number represents the system's tendency towards turbulence. For natural convection, Gr is defined as:

$Gr = \frac{\beta g(T_s - T_\infty)\ell}{U_0^2}\left(\frac{\rho_\infty U_0 \ell}{\mu}\right)^2$

$Gr = \frac{\beta g(T_s - T_\infty)\ell^3}{v^2}$ **Grashof number**

The relationship between Gr and Re can be especially useful when there are both natural and forced convection components of a problem.

If $Gr/Re^2$ is large → natural convection dominates
If $Gr/Re^2 \approx 1$ → both natural and forced convection play a role

In order to quantify the enhancement of vertical heat flux due to convection, we utilize the Nusselt number 'Nu'. The average Nusselt number is a function of the Grashof and **Prandtl 'Pr'** numbers (Pr is usually given for a material):

$\overline{Nu} = f(Gr, Pr) = f(Ra, Pr)$

Later on we will see that different forms of natural convection will have characteristic relationships between Nu and Ra such that average Nu is written:

$\overline{Nu} = c * Ra^n$ **Nusselt number**

Another handy relation is:

$$\overline{Nu} = \frac{\bar{h}\ell}{k}$$

Where $\bar{h}$ is the average heat transfer coefficient, $\ell$ is the height of the container, and k is the thermal conductivity.

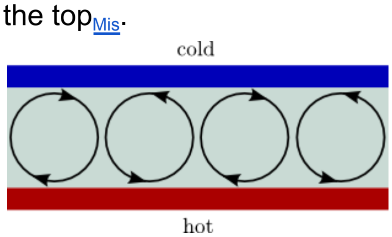
Related is the Rayleigh number 'Ra', also used to describe natural convection, giving a relation between the Grashof and Prandtl numbers. We take Ra to be:

$Ra = Gr * Pr$ **Rayleigh number relation**

# Specific Application of Natural Convection: Rayleigh-Bénard Convection

## Introduction to Rayleigh-Bénard convection

One interesting phenomenon within natural convection is the process of Rayleigh-Bénard convection. This involves a buoyancy driven flow of fluid heated from the bottom and cooled on the top[Mis].




[Fig.2 Rayleigh Benard convection pattern]

When both top and bottom are held at the same temperature, the fluid is asymptotically stable. As the bottom temperature increases, the temperature (and with it pressure and density) will vary linearly across the vertical plane of fluid. Once conduction is established, microscopic movements will spontaneously organize macroscopically into regular patterns of convection cells called Bénard cells. These are formed by the motion of the upwards moving, less dense, heated fluid in a thin layer of fluid[wiki].

 wiki

Fig.3 Bernard cell formations in gold paint dissolved in acetone.

This phenomenon can be observed commonly in miso soup or hot chocolate where the bottom is hot and cooler ambient air surrounds the opening at the top. The fluid takes on interesting characteristic motions to create such formations.

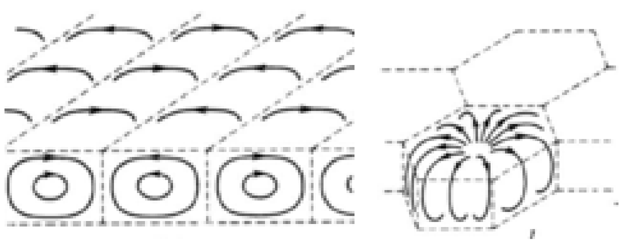
 schPedia

Fig.4 Bernard cells form through unique fluid motion.

As seen above in Fig.4, the fluid patterns create different visible cells by motions such as rolling or donut-like movement. These are only a few of the possible configurations, other variations have been observed to produce hexagonal, linear, or spiral-shaped motion profiles.

## Rayleigh-Bénard Instability

Because of the density gradient, gravity pulls cooler and denser fluid from the top to the bottom. The gravitational force is opposed by viscous damping force in the fluid. The balance between the gravitational and viscous damping forces can be represented by the Rayleigh number. In such case, we define the Rayleigh number as:

$$Ra = \frac{g\beta}{\nu\alpha}(T_b - T_u)\ell^3$$ **Rayleigh number**

Where $T_u$ is the temperature of the upper plate, $T_b$ is the temperature of the bottom plate, $\ell$ is the height of the container, $\nu$ is the kinematic viscosity, $\alpha$ is the thermal diffusivity, $g$ is the acceleration due to gravity, and $\beta$ is still the thermal expansion coefficient wiki.

The Raleigh number can be thought of as the ratio of buoyancy and viscosity forces times the ratio of momentum and thermal diffusivities wiki.

$$Ra = \left(\frac{buoyancy\ forces}{viscosity\ forces}\right)\left(\frac{thermal\ diffusivity}{momentum\ diffusivity}\right)$$

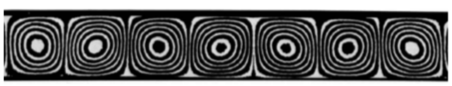
A low value indicates laminar flow and a high value indicates a turbulent flow. Below a certain value means that there is no fluid flow and heat transfer happens via conduction, not convection wiki. The critical values for this depends on the application, but for a horizontal flat plate the transition happens at $Ra \approx 10^7$ MIT.

## Small Temperature Differences

When the temperature differences between top and bottom are small, heat flows from hot to cold (bottom to top) by conduction. The fluid near the bottom hotter area is less dense and wants to rise, but the geometry constrains it. When the temperature difference is low enough, the conduction state is stable and the fluid is at rest.

## Large Temperature Differences

When the temperature differences between top and bottom are large enough, the situation becomes unstable and convection rolls set in.



[Fig.5 Convection rolls]

Some fluid rises and other fluid sinks, creating the characteristic motion profile of natural convection. At even higher temperatures, convection rolls break down and the motion becomes chaotic/turbulent.

## Equations of Motion

$$\nabla \cdot v = 0$$

$$\frac{\partial}{\partial t}T + v \cdot \nabla T = \Delta T$$

$$\frac{1}{Pr}\left(\frac{\partial}{\partial t}v + v \cdot \nabla v\right) - \Delta v + \nabla p = RaTe_z$$

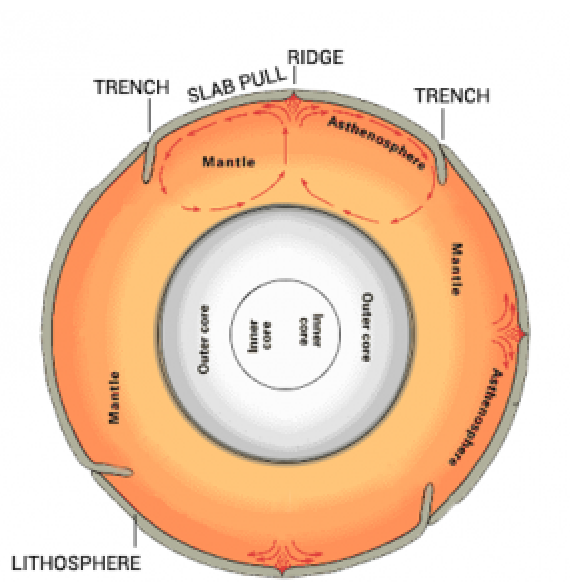
For the case of Rayleigh-Bénard convection, the Nusselt number is experimentally found to be in relation to the Rayleigh number by

$$Nu \sim Ra^{1/3}$$

Often there is a coefficient before Ra depending on the experimentally determined relationship.

# Specific Application of Natural Convection: Mantle Convection [1 hour]

Another example of natural convection is mantle convection. Mantle convection is (probably) the driving force behind plate tectonics wiki. The lower mantle is heated by the earth's core, which causes the density of the mantle closest to the core to decrease. This produces familiar natural convection behaviour where the less dense and hotter mantle rises and the more dense and cooler mantle sinks towards the earth's core. This causes the earth's lithosphere, which rests on top of the mantle to more, moving tectonic plates towards and away from each other wiki.




[Fig.6 Mantle convection and layers of the earth earthHow ]

## Calculating Rayleigh's Number for Mantle Convection

The earth's mantle is a solid, but it acts like a fluid over geological timescales wiki, i.e. 1-20 cm/year which is the speed of  mantle convection unf.

Raleigh number takes different forms for different types of heating, e.g. constant temperature diff, constant basal heat flow, or constant volumetric heat generation.

If convection is driven by a constant volumetric heat generation, Rayleigh's number can be found using the equation below.

$$Ra = \frac{g\rho_0 \alpha H D^5}{\eta k K}$$

Where $H$ is the rate of volumetric heat production, $\eta$ is the dynamic viscosity, $k$ is the thermal diffusivity, $D$ is the depth of the mantle, $K$ is thermal conductivity, and $\alpha$ is the value of thermal expansion. Note that this equation is slightly than the one used above, especially note that $\alpha$ is not the thermal diffusivity, but instead the value of thermal expansion.

If convection is driven by a constant basal heat flow, or heat flow per area, the Raleigh number can be found as below.

$$Ra = \frac{g\rho_0^{2} \alpha Q D^4}{\eta k K}$$

Where $Q$ is the heat flow at the base of the mantle.

Mantle convection is driven by a mix of both of these. Lower mantle heating by the earth's core comes from three sources: primordial heat (or heat left over from earth's creation), radioactive decay of isotopes in the core, and heat created from the moon's pull on earth[khan]. Radioactive decay, or radiogenic heat is an example of volumetric heat generation and primordial heat would contribute to a constant basal heat flow rate.

# Blog Post: Natural Convection [1.5 hours]

1. Try it yourself! Find or create an example of natural convection and take a photo/video of it in action. See Rayleigh-Bénard convection for some ideas or find some new ones of your own. Explain what is happening and how it is characteristic of natural convection.

# Natural Convection Problems [3-4 hours]

2. A 10cm flat vertical plate is held at 261℃ in 260℃ water. The thermal expansion coefficient β= 0.0022, the Prandtl number for 260℃ Pr = 0.87, and the kinematic viscosity for this same temperature $v = 0.13 \times 10^{-6}$. Given the relation between the Rayleigh and Nusselt numbers discussed for Rayleigh-Bénard, and a coefficient of 0.1 for Ra, find the Nusselt number for this situation.[Nprob]
    a. First, state the relationship between the Rayleigh and Nusselt numbers for this problem in equation form.
    b. Compute the Grashof number with the given information.
    c. Find the Nusselt number.

2. Solution:
    a. $Nu = 0.1 * Ra^{1/3}$

b. $Ra = Gr \cdot Pr$

Use $Gr = \frac{\beta g(T_s - T_\infty)\ell^3}{v^2}$

$Gr = \frac{0.0022*9.81*1*0.1^3}{(0.13*10^{-6})^2} * 0.87$= 1.11*10$^9$

c. Then use $Nu = 0.1 * Ra^{1/3}$

$Nu = 0.1 * (1.11 * 10^9)^{1/3}$ = **102.8**

A Nusselt number 100-1000 indicates turbulent flow.

3. A 2cm diameter cylinder is held at 350 K and immersed in air at 300 K. volumetric expansion coefficient βis 1/325, *v* is 1.84 x 10$^{-5}$ m$^2$/s, α is 2.62 x 10$^{-5}$m$^2$/s, and thermal conductivity *k* is 0.02m. Find the heat transfer coefficient for this scenario.
   a. Compute the Rayleigh number.
   b. Look up the Rayleigh number to find related 'c' and 'n' values for the Nusselt number relation $\overline{Nu} = c * Ra^n$. Use these to find the average Nusselt number.
   c. Determine the heat transfer coefficient. How does it compare to the rule of thumb approximation that h ≈ 10 W/m$^2$K?

3. Solution:
   a. $Ra = \frac{\beta g(T_s - T_\infty)\ell^3}{v\alpha} = \frac{(1/325)*9.81 m/s^2 * 50\ K*(0.02\ m^3)^3}{(1.84*10^{-5} m^2/s)*(2.62*10^{-5} m^2/s)} = 25,045$≈ **25,000**
   b. c=0.48, n=0.25

      $\overline{Nu} = 0.48 * Ra^{0.25}$

      Average Nusselt number $\overline{Nu} = 6$
   c. $\overline{Nu} = 6 = \frac{\bar{h}\ell}{k}$

      Heat transfer coefficient $\bar{h} = \frac{6*0.0282\ W/mK}{0.02\ m}$= 8.5 W/m$^2$K

4. The two Rayleigh numbers above are equivalent but with slightly different parameters
   a. Check the units for the Raleigh number for constant volumetric heat generation (write them out and make sure they cancel, Raleigh's number should always be dimensionless).
   b. Check the units for the basal heat flow model Raleigh number.
   c. Compare these two Raleigh numbers. Examine the terms that differ. Explain what they are and how they are equivalent across the Raleigh numbers.

4. Solution

12/12 Transport Final solutions

4. a) $$Ra = \frac{g \rho_0 \alpha H D^5}{\eta \kappa K} = \left( \frac{\left(\frac{m}{s^2}\right)\left(\frac{kg}{m^3}\right)\left(\frac{1}{K}\right)\left(\frac{W}{m^3}\right)\left(m^5\right)}{\left(\frac{kg}{m \cdot s}\right)\left(\frac{m^2}{s}\right)\left(\frac{W}{m K}\right)} \right)$$

$$= \left(\frac{kg\,W}{s^2 K}\right)\left(\frac{s^2 K}{W\,kg}\right)$$

dimensionless

b) $$Ra = \frac{g \rho \alpha Q D^4}{\eta \kappa K} = \frac{\left(\frac{m}{s^2}\right)\left(\frac{kg}{m^3}\right)\left(\frac{1}{K}\right)\left(\frac{W}{m^2}\right)\left(m^4\right)}{\left(\frac{kg}{m \cdot s}\right)\left(\frac{m^2}{s}\right)\left(\frac{W}{m K}\right)}$$

dimensionless

c) $$Ra = \frac{g \rho_0 \alpha H D^5}{\eta \kappa K} = \frac{g \rho_0 \alpha Q D^4}{\eta \kappa K}$$

$$H D^5 = Q D^4$$

$$\frac{W}{m^3} m^5 = \frac{W}{m^2} m^4$$

$$W m^2 = W m^2$$

H represents volumetric heat generation $\left(\frac{W}{m^3}\right)$ and Q represents basal per area heat flow $\left(\frac{W}{m^2}\right)$ so Because of the difference in units, the D term needs to change from $D^5$ to $D^4$.

5. Calculate these Raleigh numbers and consider what they suggest about mantle convection (the parts below will guide you through this). Consider the earth as a sphere with the radius 6371 km with a lithosphere thickness of 150 km and that the core-mantle boundary is at radius 2890 km. (Problem adapted from [UNSY](UNSY))
   a. Calculate the basal heat flow or $Q$ in $\frac{W}{m^2}$, using 12 $TW$for total heat flux $q$
   b. Calculate the radiogenic or volumetric heat production or $H$ in $\frac{W}{m^3}$, using 12 $TW$ for total heat flux $q$.
   c. Mantle convection is caused both by basal heat flow and radiogenic/volumetric heat production. How would you combine both Raleigh numbers described in the section “Calculating Rayleigh’s Number for Mantle Convection” to calculate the combined Raleigh number?
   d. If $\rho_0 = 4000\,\frac{kg}{m^3}$, $g = 10\,\frac{m}{s^2}$, $\alpha = 3 * 10^{-5}\,\frac{1}{K}$, $D = 2890\,km$, $K = 4\,\frac{W}{m*K}$, $k = 10^{-6}\,\frac{m^2}{s}$, and $\eta = 10^{22}\,Pa * s$, using the values from parts a and b and the equation from part c, calculate the combined Raleigh number. What does this number tell you about this flow? Is it more likely that it’s laminar or turbulent? You can mantle convection of this like convection from a horizontal flat plate.

5. Solution

5) a) $Q = \frac{\text{total heat flow}}{\text{surface area}} = \frac{q}{4\pi r^2}$  $r = 12$ TW $= 12 \times 10^{12}$ W

r = radius of earth's core
= 2890 km (core-mantle boundary) $= 2890 \cdot 10^3$ m

$Q = \frac{12 \text{ TW}}{4\pi (2890_{km})^2} = \frac{12 \cdot 10^{12} \text{ W}}{4\pi (2890 \cdot 10^3 \text{ m})^2}$

$= (1.143 \cdot 10^{-7})(10^6) \frac{W}{m^2}$

$= \boxed{.1143 \frac{W}{m^2}}$

b) $H = \frac{\text{heat flow}}{\text{volume}} = \frac{q}{\frac{4}{3}\pi r^3} = \frac{24 \cdot 10^{12} \text{ W}}{\frac{4}{3}\pi (2890 \cdot 10^3 \text{ m})^3}$

$= (2.374 \cdot 10^{-10})\left(\frac{10^{12}}{10^9}\right) \frac{W}{m^3} = \boxed{2.374 \cdot 10^{-7} \frac{W}{m^3}}$

c) $Ra = \frac{g \rho_0 \alpha D^4}{\eta \kappa k}(HD + Q)$  you can just add them together!

d) $Ra = \frac{(10 \, m/s)(4000 \frac{kg}{m^3})(3 \cdot 10^{-5} \frac{1}{K})(2890 \cdot 10^3 m)^4}{(10^{22} \, Pa \cdot s)(10^{-6} \frac{m^2}{s})(4 \frac{W}{mK})}$

$\cdot \left((.1143 \frac{W}{m^2})(2980 \cdot 10^3 m) + 2.374 \cdot 10^{-7} \frac{W}{m^3}\right)$

$g = 10 \, m/s$
$\rho_0 = 4000 \frac{kg}{m^3}$
$\alpha = 3 \times 10^{-5} \frac{1}{K}$
$D = 2890$ km
$k = 4 \frac{W}{mK}$
$\eta = 10^{22}$ Pa · s
$\kappa = 10^{-6} \frac{m^2}{s}$

$= \boxed{17.128 \times 10^{14}}$

flow is <u>turbulent</u>

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